# HOSSAM GHANEM

## (35) 4.2 The Mean Value Theorem (B)

| Example 8<br>39 December 14, 2006 | Let $f^{\backslash}(x) = \dfrac{1}{3+2x^2}$ for all real number $x$ and $f(1) = 0$<br>show that $\dfrac{1}{11} < f(2) < \dfrac{1}{5}$ |
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### Solution

$f$ cont. on $[1,2]$
$f$ diff. on $(1,2)$

$\therefore\ \exists\ c\ \in (1,2)$ such that $f^{\backslash}(c) = \dfrac{f(2)-f(1)}{2-1}$ I. V. T

$\dfrac{1}{3+2c^2} = \dfrac{f(2)}{2-1}$

$f(2) = \dfrac{1}{3+2c^2}$

$1 < c < 2$

$1 < c^2 < 4$

$2 < 2c^2 < 8$

$5 < 3+2c^2 < 11$

$\dfrac{1}{5} > \dfrac{1}{3+2c^2} > \dfrac{1}{11}$

$\dfrac{1}{11} < f(2) < \dfrac{1}{5}$

| Example 9<br>07/12/2011 | (4 points ) : Let $h(x)$be a differentiable function satisfying $h(15) = 123$ and $h'(x) \le 10$ for all real numbers $x$.<br>Show that $h(6) \ge 33$. |
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### Solution

$h$ cont. on $[\,6\,,15\,]$
$h$ diff. on $(\,6\,,15\,)$

$\therefore\ \exists\ c\ \in (\,6\,,15\,)$ such that $h^{\backslash}(c) = \dfrac{h(15)-h(6)}{15-6}$ I. V. T

$h^{\backslash}(c) = \dfrac{123-h(6)}{9}$

$h^{\backslash}(c) \le 10$

$\dfrac{123-h(6)}{9} \le 10$

$123 - h(6) \le 90$

$-h(6) \le 90 - 123$

$-h(6) \le -33$

$h(6) \ge 33$

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| **Example 10**<br>20 May 22, 1999 | Use the mean value theorem to prove that<br>$\lvert \tan x - \tan y \rvert \geq \lvert x - y \rvert$ for all $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

## Solution

Let $f(x) = \tan x$

$f'(x) = \sec^2 x$

Let $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $x < y$

$f$ cont. on $[x, y]$

$f$ diff. on $(x, y)$

$\therefore\ \exists\ c \in (x, y)$ such that $f'(c) = \dfrac{f(y) - f(x)}{y - x}$ I.V.T

$\sec^2 c = \dfrac{\tan y - \tan x}{y - x}$

$\lvert \sec^2 c \rvert \geq 1$

$\left| \dfrac{\tan y - \tan x}{y - x} \right| \geq 1$

$\lvert \tan y - \tan x \rvert \geq \lvert y - x \rvert$

or

$\lvert \tan y - \tan x \rvert \geq \lvert x - y \rvert$

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| **Example 11**<br>2 May 20, 1993 | Use the M.V.T to show that $\lvert \sin^2 a - \sin^2 b \rvert \leq \lvert a - b \rvert$ for all real numbers $a$ and $b$ |

## Solution

Let $f(x) = \sin^2 x$ , $a < b$

$f$ cont. on $[a, b]$

$f$ diff. on $(a, b)$

$\therefore\ \exists\ c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$f'(x) = 2 \sin x \cos x = \sin 2x$

$f'(c) = \sin 2c$

$\sin 2c = \dfrac{\sin^2 b - \sin^2 a}{b - a}$

$\lvert \sin 2c \rvert \leq 1$

$\left| \dfrac{\sin^2 b - \sin^2 a}{b - a} \right| \leq 1$

$\lvert \sin^2 b - \sin^2 a \rvert \leq \lvert b - a \rvert$

or

$\lvert \sin^2 b - \sin^2 a \rvert \leq \lvert a - b \rvert$

| Example 12<br>14 January 6, 1996 | Use the mean value theorem to show that<br>If $f^{\backslash}(x) > 0$ on any interval then $f$ is increasing |
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## Solution

Let $f$ diff. function and let $a, b \in R$ such that $a < b$
$f$ cont. on $[a, b]$
$f$ diff. on $(a, b)$

$\therefore \ \exists \ c \in (a, b)$ such that $f^{\backslash}(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$f^{\backslash}(x) > 0 \quad \rightarrow \quad \therefore f^{\backslash}(c) > 0$

$\dfrac{f(b) - f(a)}{b - a} > 0$

$f(b) - f(a) > 0$

$f(b) > f(a)$

$\therefore f$ is increasing

| Example 13<br>12 January 1995 | Use mean value theorem to prove that<br>if $f^{\backslash}(x) = 0$ on an interval I then $f(x)$ is constant |
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## Solution

Let $f$ diff. function and let $a, b \in R$ such that $a < b$
$f$ cont. on $[a, b]$
$f$ diff. on $(a, b)$

$\therefore \ \exists \ c \in (a, b)$ such that $f^{\backslash}(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$\because \ f^{\backslash}(x) = 0$

$\therefore \ f^{\backslash}(c) = 0$

$\dfrac{f(b) - f(a)}{b - a} = 0$

$f(b) - f(a) = 0$

$f(b) = f(a)$

$\therefore f$ is constant

| Example 14<br>36 January 17, 2010 | (4 pts.) Let $f$ be a function such that $f(0) = 0$ and $f'(c) < 2$ $\forall c \in \mathcal{R}$. Show that $f(x) < 2x \ \forall \ x > 0$ |
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## Solution

Let $f$ diff. function
$f$ cont. on $[0, x]$
$f$ diff. on $(0, x)$

$\therefore \ \exists \ c \in (0, x)$ such that $f^{\backslash}(c) = \dfrac{f(x) - f(0)}{x - 0}$ I.V.T

$f^{\backslash}(c) = \dfrac{f(x) - 0}{x}$

$\therefore \ f^{\backslash}(c) < 2$

$\dfrac{f(x)}{x} < 2$

$f(x) < 2x$

# Homework

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| 1 | 24 July 20, 2000 A<br>Use the mean value theorem to show that if $u$ and $v$ are any real numbers , then<br>$\lvert\cos^7 u - \cos^7 v\rvert \le 7\lvert u - v\rvert.$ |
| 2 | 4 December 15, 1994<br>show that $\lvert\sin(2b) - \sin(2a)\rvert \le 2\lvert b - a\rvert$ for any real numbers $a$ and $b$ |
| 3 | 6 January 6, 1993<br>Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$ |
| 4 | Use the M.V.T to show that $\lvert\sin a - \sin b\rvert \le \lvert a - b\rvert$ for all real numbers $a$ and $b$ |
| 5 | Use the M.V.T to show that<br>$\lvert\tan a - \tan b\rvert \ge \lvert a - b\rvert$ for all real numbers $a , b \in \left(-\frac{\pi}{2} , \frac{\pi}{2}\right)$ |
| 6 | 50 22 December 2010<br>Let $f$ be continuous on $[\, a , b \,]$ with $f'(x) < 0$ for all $x \in (a , b\,)$.<br>Show that $f(b) < f(a)$ |
| 7 | 6 May 9, 1996<br>show that $\lvert\cos^5 u - \cos^5 v\rvert \le \frac{5}{2}\lvert u - v\rvert$ For any real numbers $u$ and $v$ |
| 8 | 35 December 16, 2004<br>Suppose that $f$ is continuous on $[\, a \, , b \,]$ and $f^{\backslash}(x) < 0$ for every $x \in (\, a \, , b\,)$ .<br>Use The Mean Value Theorem to show that $f(b) < f(a)$. |
| 9 | 33 January 20, 2009<br>Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem, of the function $f$ on the interval $[-1 , 2\,]$ , where $f(x) = x^3 - 2x$ |
| 10 | 30 May 15th, 2003<br>Let $f$ be a differentiable function on $[\,1 , 3\,]$ with $f(1) = 3$ and $f(3) = 1$. Show that the graph of $f$ admits a tangent line at $c \in (\,1 , 3\,)$ parallel to the line of the equation:<br>$x + y - 4 = 0.$ |
| 11 | 48 Sunday 9 May 2010<br>Use the Mean Value Theorem to show that<br>$\tan(3/2) - \tan(1/2) > 1.$ [2 marks] |

# Homework

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| 12 | 18 December 3, 1998<br>Use The Mean Value Theorem to prove that<br>If $\lvert f^{\backslash}(x) \rvert \leq M$ for all $x \in (a\,,b)$ , and if $x_1$ and $x_2$ are any two point in $(a\,,b)$ with $x_1 < x_2$ then $\lvert f(x_2) - f(x_1) \rvert \leq M \lvert x_2 - x_1 \rvert$ |
| 13 | 13 February 19, 1995<br>For the function<br>$f(x) = Ax^2 + Bx + C$ With $A \neq 0$ on the interval $[a\,,b\,]$<br>Show that the number $c$ in $(a\,,b\,)$ determined by the mean value theorem is the midpoint of the interval |
| 14 | Suppose $f$ is function continues on $[\,a\,,b\,]$ , differentiable on $(\,a\,,b\,)$ and satisfies $f(a) < f(b)$ Show that there exist a value $c \in (a\ ,\ b)$ such that $f^{\backslash}(c)$ positive |
| 15 | Let $f(x) = x^5 + 16x + 10$ .<br>Use the Mean Value Theorem to show that $\forall\, a\,,b \in \mathcal{R}\ ,\ a \neq b$ then $f(a) \neq f(b)$ |

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| **12**<br>18 December 3, 1998 | Use The Mean Value Theorem to prove that<br>If $\left\|f^{\backslash}(x)\right\| \leq M$ for all $x \in (a\,,b)$ , and if $x_1$ and $x_2$ are any two point in $(a\,,b)$ with $x_1 < x_2$ then $\|f(x_2) - f(x_1)\| \leq M\|x_2 - x_1\|$ |

## Solution

$f$ cont. on $[a,b]$ $\Rightarrow \therefore f$ cont. on $[x_1, x_2]$

$f$ diff. on $(a,b)$ $\Rightarrow \therefore f$ diff. on $(x_1, x_2)$

$\therefore\ \exists\ c\ \in (x_1, x_2)$ such that $f^{\backslash}(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ I.V.T

$|f^{\backslash}(c)| \leq M$

$\left|\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}\right| \leq M$

$\dfrac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq M$

$|f(x_2) - f(x_1)| \leq M\,|x_2 - x_1|$

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| **13**<br>13 February 19, 1995 | For the function<br>$f(x) = Ax^2 + Bx + C$ With $A \neq 0$ on the interval $[a\,,b\,]$<br>Show that the number $c$ in $(a\,,b\,)$ determined by the mean value theorem is the midpoint of the interval |

## Solution

$f$ cont. on $[a,b]$

$f$ diff. on $(a,b)$

$\therefore\ \exists\ c\ \in (a,b)$ such that $f^{\backslash}(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$f^{\backslash}(x) = 2Ax + B$

$f^{\backslash}(c) = 2Ac + B$

$2Ac + B = \dfrac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$

$2Ac + B = \dfrac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a}$

$2Ac + B = \dfrac{A(b^2 - a^2) + B(b - a)}{b - a}$

$2Ac + B = \dfrac{A(b - a)(b + a) + B(b - a)}{b - a}$

$2Ac + B = A(b + a) + B$

$2Ac = A(b + a)$

$2c = b + a$

$c = \dfrac{b + a}{2}$

$\therefore c$ is midpoint of interval $(a,b)$

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| 14 | Suppose $f$ is function continues on $[\, a\,, b\,]$ , differentiable on $(\, a\,, b\,)$ and satisfies $f(a) < f(b)$ Show that there exist a value $c \in (a\ ,\ b)$ such that $f^{\backslash}(c)$ positive |

## Solution

$f$ cont. on $[a, b]$

$f$ diff. on $(a, b)$

$\therefore\ \exists\ c\ \in (a, b)$ such that $f^{\backslash}(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$\because\ f(a) < f(b)$

$f(b) - f(a)\ > 0$

$\dfrac{f(b) - f(a)}{b - a}\ > 0$

$\therefore\ f^{\backslash}(c)\ > 0$

$\therefore f$ is positive

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| 15 | Let $f(x) = x^5 + 16x + 10$ . Use the Mean Value Theorem to show that $\forall\, a\,, b\ \in \mathcal{R}\ ,\ a \neq b$ then $f(a) \neq f(b)$ |

## Solution

Let $a < b$

$f$ is polynomial

$f$ cont. on $[a, b]$

$f$ diff. on $(a, b)$

$\therefore\ \exists\ c\ \in (a, b)$ such that $f^{\backslash}(c) = \dfrac{f(b) - f(a)}{b - a}$ I.V.T

$f^{\backslash}(x) = 5x^4 + 16$

$f^{\backslash}(c) = 5x^4 + 16\ \neq 0$

$\therefore \dfrac{f(b) - f(a)}{b - a}\ \neq 0$

$f(b) - f(a)\ \neq 0$

$f(b) \neq f(a)$